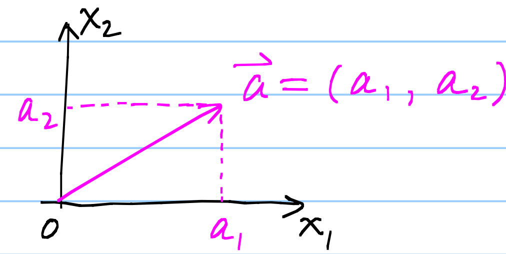
Note Title 7/12/2011

Math Background:

1. From Linear Algebra.

— Vectors



Magnitude (length):

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$$

In n-D, $\vec{a} = (a_1, a_2, ---, a_n)$.

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

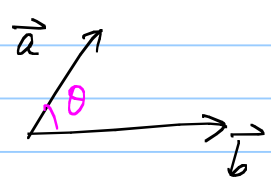
Given 2 vectors, $\vec{a} = (a_1, a_2, ---, a_n)$,

$\vec{b} = (b_1, b_2, ---, b_n)$.

The DOT PRODUCT between them is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + --- + a_n b_n$$

An application of Dot Product:



$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}$$

- Matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 4 & 5 & 2 \\ 6 & 7 & 10 \end{bmatrix}_{4\times 3}$$

- Matrix Addition (& Subtraction):

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{bmatrix}.$$

- Matrix Multiplication

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{bmatrix}.$$

$AB = ?$

Key Idea:

Take each row of A,
each column of B. $\Big\}$ dot product!

First, a matrix times a column vector:

$$A\vec{b} = \underset{2\times 3}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

$$= \underset{2\times 1}{\begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{bmatrix}}$$

Note: A matrix times a vector is a vector.

Next, a matrix times a matrix:

$$AB = \underset{2\times 3}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{33} \end{bmatrix}} = C$$

Must match

$$= \underset{2\times 2}{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}$$

$$c_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

$$= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

Pick 2nd column from B

$$c_{12} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}$$

Pick 1st row from A

$$= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

$$c_{21} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

$$= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

$$c_{22} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$$

eg.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{bmatrix}_{3\times 3}$$

$$AB = \begin{bmatrix} 6 & 20 & 8 \\ 10 & 32 & 12 \end{bmatrix}$$

$$BA = ?$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}_{2\times 3} = \text{nonsense!}$$

Must match!

Q: How about $\frac{A}{B} = ?$

In the world of numbers:

for any number $a \neq 0$,

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

In the world of matrices,

Given an $n \times n$ matrix A,

if you can find another matrix B,

such that

$$AB = BA = I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{n \times n}.$$

Identity matrix.

then B is the inverse matrix of A.

Notation: $B = A^{-1}$.

For 2×2 matrices, a very quick way —

eg. $A = \begin{bmatrix} 3 & 1 \\ 7 & 9 \end{bmatrix}$, $A^{-1} = ?$

$$A^{-1} = \begin{bmatrix} 9/20 & -1/20 \\ -7/20 & 3/20 \end{bmatrix}.$$

Check: $\frac{1}{20}\begin{bmatrix} 9 & -1 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 9 \end{bmatrix}$

$$= \frac{1}{20}\begin{bmatrix} 27-7 & 9-9 \\ -21+21 & -7+27 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Secret ?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^{-1} = ?$$

$$A^{-1} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Concept: determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

How about the 3×3 case ?

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ?$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$$

$$- a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

A better way: expand along any row or any column.

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Now, matrix inverse for 3×3 case:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} A^*$$

Concept: minor

Pick a_{ij}, the minor of a_{ij} is the determinant consisting of the remaining numbers by deleting the i^{th} row and j^{th} column where a_{ij} lies.

The minor of a_{11} is $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$,

" a_{22} " $A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$.

The Cofactor:

$$C_{ij} = (-1)^{i+j} A_{ij}$$

$$A^* = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

In matrix jargon: transposed.

If $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$$A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Here, in our inverse matrix,

$$A^* = Cof(A)^T$$

cofactor

$$\boxed{A^{-1} = \frac{1}{det} Cof(A)^T}$$

eg. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 4 & 5 \end{bmatrix}$. $A^{-1} = ?$

Soln: $det(A) = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 0 & 3 \\ 4 & 5 \end{vmatrix}$

$+ (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}$

$+ (-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$

$= -12 - (10-3) + 8 = -11.$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 4 & 5 \end{vmatrix} = -12$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = -7$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 8$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 4 & 5 \end{vmatrix} = -1$$

$C_{22} = 4$ $\qquad$ $C_{23} = -3$

$C_{31} = 3$ $\qquad$ $C_{32} = -1$

$C_{33} = -2$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Cof}(A)^T$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \frac{1}{-11} \begin{bmatrix} -12 & -1 & 3 \\ -7 & 4 & -1 \\ 8 & -3 & -2 \end{bmatrix}.$$

2. From Calculus:

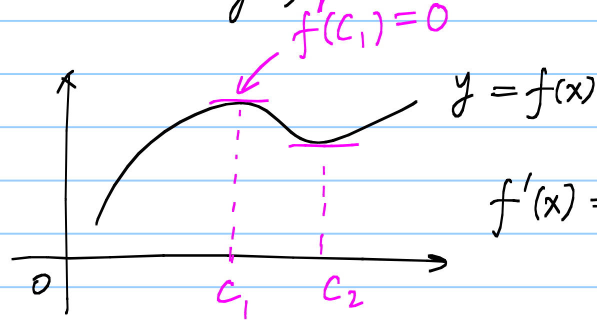
- Derivative:

Given a function $y = f(x)$,

the derivative of $f(x)$ is

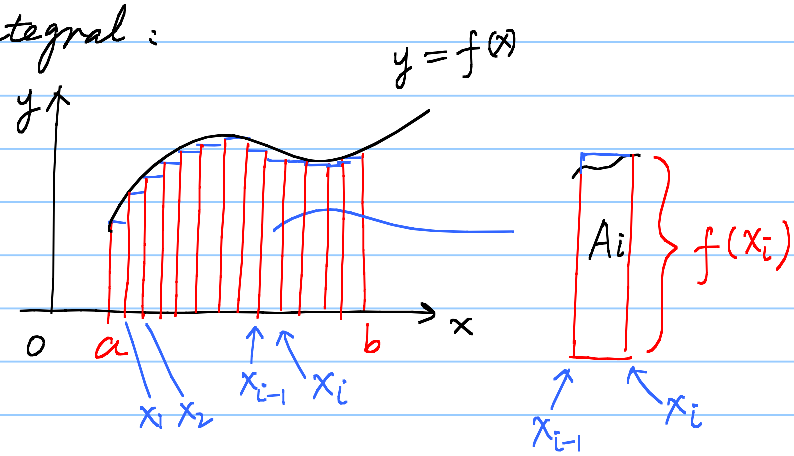
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Application: to solve optimization problems. (to maximize or minimize something).



$f'(x)$ = slope of tangent at x.

— Integral:



$\Delta x = x_i - x_{i-1}$

$$A_i \approx \text{height} \times \text{base}$$

$$= f(x_i)\,\Delta x,$$

$$A \approx A_1 + A_2 + \cdots + A_n$$

$$= \sum_{i=1}^{n} A_i$$

$$= \sum_{i=1}^{n} f(x_i)\,\Delta x$$

How to make the approximation exact?

Take the limit, let $n \to \infty$.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

$$= \int_a^b f(x)\, dx$$

3. From Probability:

Sample Point = any possible outcome.

S = Sample Space = ALL possible outcome.
(a set).

E = Event = Any subset of S.

$$P(E) = \frac{\#\text{ of sample point in } E}{\#\text{ of } \text{〃} \quad \text{〃} \text{ in } S}.$$

eg. Toss a die: $S = \{1, 2, 3, 4, 5, 6\}$

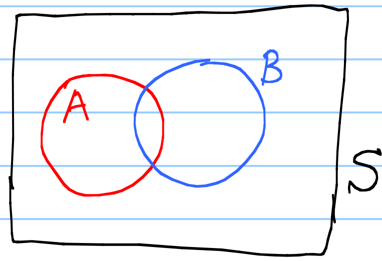
E = "less than 3" $= \{1, 2\}$

$$P(E) = \frac{2}{6} = \frac{1}{3}.$$

Conditional Probability:

$P(A|B)$ = Probability of the event A occurs, given the condition

B has already occurred.



intersect

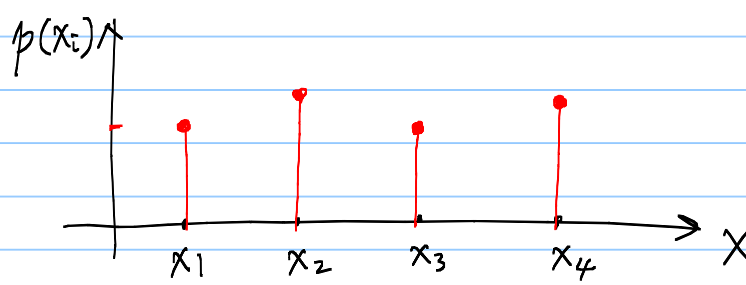
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Key Concept: Mean & Variance.

constant $\longleftrightarrow$ variable

fixed changing.

Random Variable: a variable which can take some possible values with certain probability.



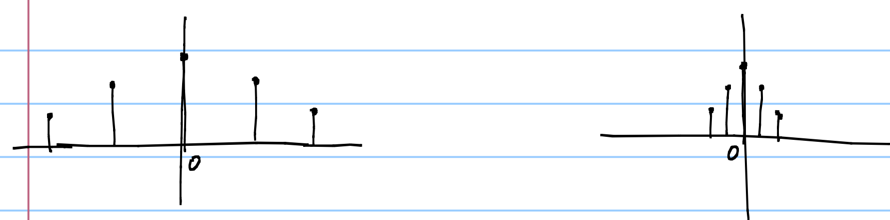
a random variable X can take possible values $x_1, x_2, ---, x_n$ with probability $p_1, p_2, ---, p_n$.

The mean (expected value) of X is

$$\mu = E[X] = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$
$$= \sum_{i=1}^{n} x_i p_i$$

Variance:

$$\sigma^2 = Var(X) = E[(X-\mu)^2]$$
$$= \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$



Given 2 random variables X and Y,

How to describe their relative relationship?

Covariance

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$$

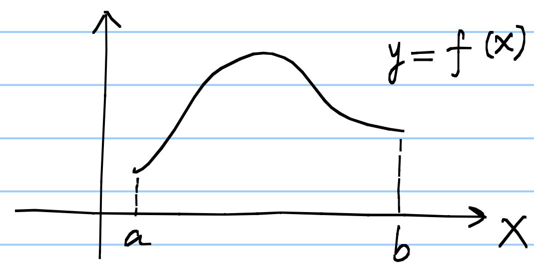
For discrete random variables,

$$E[X] = \sum_{i=1}^{n} x_i p_i$$

For Continuous random variables

Need to know the

probability density function (PDF)

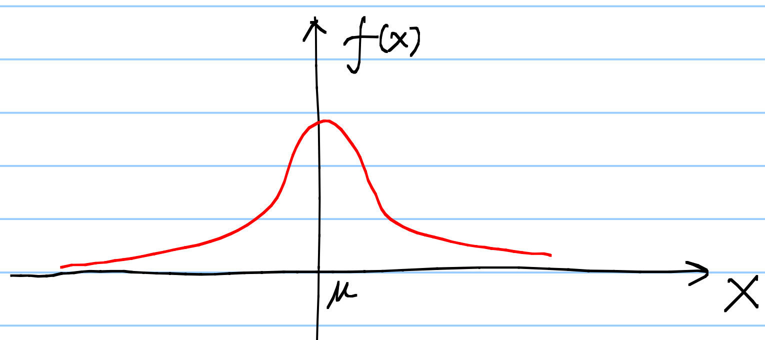


$$\mu = E[X] = \int_a^b x f(x)\, dx$$

$$\sigma^2 = E[(X-\mu)^2] = \int_a^b (x-\mu)^2 f(x)\, dx .$$

— Gaussian Distribution

(Normal Distribution)



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} .$$

A normal distributed random variable

X with mean μ, variance σ^2.

Notation: $X \sim N(\mu, \sigma^2)$.

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