Kalman Filter

Note Title 7/12/2011

Filter for water : to get rid of impurity.

Digital Filter : to get rid of noise,
recover the original signal.

Thiele, Swerling (1958)

Stratonovich (1959)

Kalman (1960)

Simple Examples (for motivation).

e.g. We try to estimate a mean of some parameter (say, wind's speed).

You have recorded $x_1, x_2, \cdots, x_n$.

$$\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

the average from first n points.

When you get the next new number,

$$\bar{x}_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i \quad \leftarrow$$

wasted due to repeated calculation

$$= \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i + x_{n+1}\right)$$

$$= \frac{1}{n+1}\left(n \cdot \frac{1}{n}\sum_{i=1}^{n} x_i + x_{n+1}\right)$$

$$= \frac{1}{n+1}\left(n\,\overline{x}_n + x_{n+1}\right)$$

$$= \frac{n}{n+1}\overline{x}_n + \frac{1}{n+1}x_{n+1}$$

Insight: Use the old estimate $\overline{x}_n$,

and also take the newly updated

info x_{n+1} into consideration,

$\longrightarrow$ get the new estimate $\overline{x}_{n+1}$.

using a weighted average

between $\overline{x}_n$ and x_{n+1}.

eg. We need to estimate the room temperature.

Assistant 1: variance $\sigma_1^2 = \pm 2$.

Assistant 2: " $\sigma_2^2 = \pm 5$.

How to use both estimates to improve

your overall estimate?

Key idea: again, weighted average.

Final Estimate of temperature $\hat{T}$

$$\hat{T} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\hat{T}_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\hat{T}_2$$

$$= \frac{5}{2+5}\hat{T}_1 + \frac{2}{2+5}\hat{T}_2$$

Kalman Filter (First Encounter) .

— Introduced a "state" of the system .

x_k = the state at time k .

State Equation :

$$\underbrace{x_{k+1}}_{\text{next state}} = A\underbrace{x_k}_{\text{current state}} + B\underbrace{u_k}_{\text{your control}} + \underbrace{w_k}_{\text{process noise}} \qquad (1)$$

Output Equation :

$$\underbrace{z_k}_{\text{current output}} = C\underbrace{x_k}_{\text{current state}} + \underbrace{m_k}_{\text{measurement noise}} \qquad (2)$$

Assumption :

$w_k \sim N(0, Q)$,

$m_k \sim N(0, R)$.

In general, x_k, u_k, w_k, m_k — vectors.

A, B, C — matrices

Coviance matrix:

$$Q = E[w_k w_k^T]$$

$$R = E[m_k m_k^T].$$

The Key Idea in KF:

$$\boxed{\begin{array}{c}\text{Time Update}\\ \text{(Predict)}\end{array}} \rightleftarrows \boxed{\begin{array}{c}\text{Measurement Update}\\ \text{(Correct)}\end{array}}$$

Notation:

x_k = the true current state

$\hat{x}_k^-$ = <u>a priori</u> estimate of x_k only based on the knowledge of the process priori to step k.

$\hat{x}_k$ = final decision on the estimate of x_k based on both $\hat{x}_k^-$ and new measurement.

Estimate Error :

$$e_k^- = x_k - \hat{x}_k^- \quad (\text{a priori error})$$

$$e_k = x_k - \hat{x}_k \quad (\text{a posteriori error})$$

Covariance of error :

$$P_k^- = E[e_k^- e_k^{-T}]$$

$$P_k = E[e_k e_k^T]$$

KF :

(1) $\hat{x}_k^- = A\hat{x}_{k-1} + B u_{k-1}$

(2) $P_k^- = AP_{k-1}A^T + Q$

Time Update

Kalman Gain — the weight to get new average

(3) $K_k = P_k^- C^T (C P_k^- C^T + R)^{-1}$

(4) $\hat{x}_k = \hat{x}_k^- + K_k (z_k - C\hat{x}_k^-)$

(5) $P_k = (I - K_k C) P_k^-$

Measurement Update.

eg. We want to estimate a voltage.

Time (ms)	1	2	3	4	---	10
Value (V)	0.39	0.50	0.48	0.29	---	0.45

Suppose: no control input $u_k = 0$.

no process noise $w_k = 0$.

only measurement noise m_k. ✓

From experience, we knew

$$m_k \sim N(0, \sigma^2),$$

$$\sigma = 0.1\ V.$$

x_k = the voltage at k.

$$x_k = A x_{k-1} + B \underbrace{u_{k-1}}_{=0} + \underbrace{w_{k-1}}_{=0}.$$

$$= x_{k-1} \qquad A = 1$$

$$z_k = C x_k + m_k \qquad C = 1.$$

$$= x_k + m_k.$$

my reading

actual voltage

my measurement error.

eg. To detect a robot's position: p_k.

Need a state vector:

$$x_k = \begin{bmatrix} p_k \\ v_k \end{bmatrix}.$$

From physics: (suppose we measure every T seconds).

$$v_{k+1} = v_k + T\underbrace{a_k}_{\text{acceleration.}} \quad \text{(ideal case)}.$$

$$v_{k+1} = v_k + Ta_k + \underbrace{\tilde{v}_k}_{\text{velocity noise, from wind, bumps.}} \quad \text{(in reality)}$$

(i) $$p_{k+1} = p_k + Tv_k + \frac{a_k}{2}T^2 + \underbrace{\tilde{p}_k}_{\text{position noise.}}$$

(ii) $$v_{k+1} = v_k + Ta_k + \tilde{v}_k$$

$$\begin{bmatrix} p_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} p_k + Tv_k \\ 0.p_k + v_k \end{bmatrix} + \begin{bmatrix} \frac{1}{2}T^2 a_k \\ Ta_k \end{bmatrix} + \begin{bmatrix} \tilde{p}_k \\ \tilde{v}_k \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} p_{k+1} \\ v_{k+1} \end{bmatrix}}_{x_{k+1}} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p_k \\ v_k \end{bmatrix}}_{x_k} + \underbrace{\begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} a_k}_{Bu_k} + \underbrace{\begin{bmatrix} \tilde{p}_k \\ \tilde{v}_k \end{bmatrix}}_{w_k}$$

$$x_{k+1} = A x_k + B u_k + w_k .$$

Exactly the KF state equation!

How about the output equation?

We just want p_k as the output.

$$z_k = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} p_k \\ v_k \end{bmatrix}}_{x_k} + m_k$$

$$z_k = C x_k + m_k .$$

measured position.

measurement error

Time Update
(Predict)

(1) Predict the state ahead

$$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_{k-1}$$

(2) Predict the error covariance ahead

$$P_k^- = AP_{k-1}A^T + Q$$

Measurement Update
(Correct)

(3) Compute the Kalman gain (as a weight)

$$K_k = P_k^- C^T (C P_k^- C^T + R)^{-1}$$

(4) Adjust the estimate based on new measurement

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - C\hat{x}_k^-)$$

(5) Update the error covariance

$$P_k = (I - K_k C)P_k^-$$